Malaysian Journal of Mathematical Sciences 16(1): 87–103 (2022)
https://doi.org/10.47836/mjms.16.1.08

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

Modified HAM for solving linear system of Fredholm-Volterra Integral Equations

Eshkuvatov, Z. K. *[1,3], Ismail, Sh. [2], Mamatova, H. X. [1,3], Viscarra, D. S. [4], and Aloev, R. D. [3]

[1]*Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terangganu, Malaysia*
[2]*Faculty of Science and Technology, Universiti Sains Islam Malaysia, Malaysia*
[3]*Faculty of Applied Mathematics and Intellectual Technologies, National University of Uzbekistan, Uzbekistan*
[4]*Departamento de Matemáticas, Universidad Simón I. Patiño, Bolivia*

E-mail: zainidin@umt.edu.my
**Corresponding author*

Received: 8 March 2020
Accepted: 15 July 2021

Abstract

This paper considers systems of linear Fredholm-Volterra integral equations using a modified homotopy analysis method (MHAM) and the Gauss-Legendre quadrature formula (GLQF) to find approximate solutions. Standard homotopy analysis method (HAM), MHAM, and optimal homotopy asymptotic method (OHAM) are compared for the same number of iterations. It is noted from the chosen examples that MHAM with GLQF is comparable with standard HAM and OHAM. In all cases, MHAM with GLQF approaches exact solutions, where residual rapidly converges to zero when the number of iterations and quadrature nodes increases. The HAM developed in this paper is better than the HAM developed by Shidfar & Molabahrami in "Solving a system of integral equations by an analytic method".

Keywords: homotopy analysis method, Gauss-Legendre quadrature formula, approximate solutions, convergence.

1 Introduction

The homotopy analysis method (HAM) was first introduced by Liao [9] in his PhD thesis, where he successfully applied it to nonlinear problems. A systematic and clear exposition on HAM is given in Liao's work [2]. After the publication of that work, a number of researchers have successfully applied this method to various nonlinear problems in science and engineering. For example, see [10] discussed the non-linear problems, [11, 7] discussed fluid dynamics, [12] discussed the non-linear problem and comparison between HPM and HAM, [1] elaborated on KdV equations, [14] discussed non-linear differential problem, and [3] elaborated on engineering and biology. The HAM contains an auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution. Moreover, employing the so-called $\hbar$-curve, it is easy to find the valid regions of $\hbar$ to gain a convergent series solution. The main aim is to analyze the HAM from a mathematical point of view utilizing some basic nonlinear and linear problems of the system of integral equations and hence show a presentable comparison by obtaining the exact solution of the homotopy equations. On the other hand, employing the convergence-control parameter can always be avoided in the HAM. Thus, HAM is an explicit semi-analytic solution of linear and nonlinear problems.

In numerical analysis, a quadrature rule approximates the definite integral of a function, usually stated as a weighted sum of function values at specific points within the domain of integration. An n-point Gaussian quadrature rule, named after Carl Friedrich Gauss, is a quadrature rule constructed to give exact results for polynomials of degree $2n-1$ or less, employing a suitable choice of nodes $\{x_i\}$ and weights $\{w_i\}$, for $i \in \{1, 2, \ldots, n\}$. Legendre polynomials $P_n(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with unit weight. The Gauss-Legendre quadrature formula (GLQF) is a special case of Gaussian quadrature that allows us to approximate a function with known asymptotic behavior at the edges of the integration interval. Applications of the GLQF can be found in many references. For example, [4] derived error estimates for the Gauss Legendre and the Gauss-Chebyshev quadrature formulas for analytic functions, [5] discussed Gauss-Legendre quadrature formula (GLQF) extended to the kernel integrals of two variables and proved the convergence of GLQF, while [8] discussed numerous methods on integration problems including GLQF.

Consider a general system of linear Fredholm-Volterra integral equations (FVIEs) of the form

$$\bar{U}(x) = \bar{G}(x) + \lambda_1 \int_a^b \bar{K}_1(x,t)\bar{U}(t)\,dt + \lambda_2 \int_a^x \bar{K}_2(x,t)\bar{U}(t)\,dt, \tag{1}$$

where

$$\begin{aligned} &\bar{U}(x) = \left(u_1(x), u_2(x), \ldots, u_n(x)\right), \\ &\bar{G}(x) = \left(g_1(x), g_2(x), \ldots, g_n(x)\right), \\ &\bar{K}_r(x,t) = \left(k_{r,ij}(x,t)\right), r \in \{1,2\}, i,j \in \{1,2,\ldots,n\}, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}$ denotes the independent variable; $g_i(x)$ are known analytic functions, and $u_i(x)$ are unknown functions to be determined. For $g_i(x) = 0$, (1) reduces to the homogeneous equation, and the theory of eigenvalues and eigenfunctions can be applied.

Ghazanfari & Yari [6] considered (1) with $\lambda_2 = 0$, and applied optimal homotopy asymptotic method (OHAM) to obtain an approximate solution. This work showed the effectiveness of OHAM for (1) and provided easy tools to control the convergence region of approximating solution series, wherever necessary. The results of OHAM were compared with the homotopy perturbation method and the Taylor series expansion method. Shidfar & Molabahrami [13] proposed HAM for solving systems of linear and nonlinear Volterra and Fredholm integral equations. Using HAM, it is possible to find the exact solution or approximate solution for the problem. The efficiency of the approach was illustrated by applying the procedure to several examples.

This paper is arranged in the following manner. In Section 2, we present the standard HAM and modified HAM. Section 3 describes the implementation of standard HAM and modified HAM for the problem (1)–(2). In Section 4, a GLQF is developed for kernel integration. Section 5 deals with examples and shows a comparison of the proposed method with standard HAM and OHAM. Finally, some conclusions and acknowledgements are given in Section 6.

2 Analysis of the HAM

To apply HAM to the i-th equation of (1), let us rewrite it as follows:

$$u_i(x) = g_i(x) + \sum_{j=1}^{n}\left[\lambda_1 \int_a^b k_{1,ij}(x,t)u_j(t)\,dt + \lambda_2 \int_a^x k_{2,ij}(x,t)u_j(t)\,dt\right], i \in \{1,2,...,n\}. \tag{3}$$

The operator form of (3) is

$$N_i\left[u_i(x)\right] = g_i(x), i \in \{1,2,...,n\}, \tag{4}$$

where

$$N_i\left[u_i(x)\right] = u_i(x) - \sum_{j=1}^{n}\left[\lambda_1 \int_a^b k_{1,ij}(x,t)u_j(t)\,dt + \lambda_2 \int_a^x k_{2,ij}(x,t)u_j(t)\,dt\right].$$

In general, the operator N_i in (4) can be nonlinear or singular.

For the sake of clarity, we will first present a brief description of the standard HAM. From it, we will derive the description of the algorithm of modified HAM (MHAM).

To generalize the traditional homotopy method, Liao [9] constructed the so-called *zeroth-order deformation equations*,

$$(1-q)L\left[\phi_i(x,q) - u_{i,0}(x)\right] = q\hbar\left\{N_i\left[\phi_i(x,q)\right] - g_i(x)\right\}, \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_{i,0}(x)$ are initial guesses of exact solutions $u_i(x)$, and $\phi_i(x,q)$ are unknown functions.

Obviously, when $q = 0$ and $q = 1$, then

$$\phi_i(x,0) = u_{i,0}(x) \quad \text{and} \quad \phi_i(x,1) = u_i(x)$$

hold, respectively. Thus, as q increases from 0 to 1, the function $\phi_i(x,q)$ varies from the initial guess $u_{i,0}(x)$ to the solution $u_i(x)$—this is called *deformation*. Indeed, expanding $\phi_i(x,q)$ in Taylor series with respect to q, one has

$$\phi_i(x,q) = u_{i,0}(x) + \sum_{k=1}^{\infty} u_{i,k}(x) q^k, \tag{6}$$

where

$$u_{i,k}(x) = \frac{1}{k!} \cdot \left. \frac{\partial^k \left[\phi_i(x,q)\right]}{\partial q^k} \right|_{q=0}. \tag{7}$$

As long as the auxiliary linear operator L, the initial guess $u_{i,0}$, the auxiliary parameter $\hbar$, and the auxiliary functions are properly chosen, the series (6) converges at $q=1$. Then, one has

$$\phi_i(x,1) = u_i(x) = u_{i,0}(x) + \sum_{k=1}^{\infty} u_{i,k}(x), \tag{8}$$

which must be one of the solutions of the original system of nonlinear equations (4).

Based on (7), the governing equation can be deduced from the zeroth-order deformation equations (5). To do this, let us define the vectors

$$\bar{u}_{i,n} = \left(u_{i,0}(x), u_{i,1}(x), \ldots, u_{i,n}(x)\right).$$

Differentiating (5) m times with respect to the embedding parameter q, then setting $q=0$, and finally dividing by $m!$, we have the so-called m-th order deformation equation,

$$L\left[u_{i,m}(x) - \chi_m u_{i,m-1}(x)\right] = \hbar R_{i,m}\left(\bar{u}_{i,m-1}\right), \tag{9}$$

where

$$R_{i,m}\left[\bar{u}_{i,m-1}(x)\right] = \frac{1}{(m-1)!} \cdot \left. \frac{\partial^{m-1}\left\{N_i[\phi_i(x,q)] - g_i(x)\right\}}{\partial q^{m-1}} \right|_{q=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \le 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{11}$$

The modified form of the HAM can be obtained by the assumption that the function $g_i(x)$ can be split into n terms, namely

$$g_i(x) = s_{i,0}(x) + s_{i,1}(x) + \cdots + s_{i,n}(x).$$

Expanding $g_i(x) = \varphi_i(x,q)$ into powers of the embedding parameter q, we have

$$\varphi_i(x,q) = s_{i,0}(x)q^0 + s_{i,1}(x)q^1 + \cdots + s_{i,n}(x)q^n = \sum_{j=0}^{n} q^j s_{i,j}(x). \tag{12}$$

From (5) and (12), the new zeroth-order deformation equation has the form

$$(1-q)L\left[\phi_i(x,q)-u_{i,0}(x)\right]=q\hbar\left\{N_i\left[\phi_i(x,q)\right]-\varphi_i(x,q)\right\},$$

and the m-th order deformation equation is

$$L\left[u_{i,m}(x)-\chi_m u_{i,m-1}(x)\right]=\hbar R_{i,m}\left(\bar{u}_{i,m-1}\right), \tag{13}$$

where

$$R_{i,m}\left[\bar{u}_{i,m-1}(x)\right]=\frac{1}{(m-1)!}\cdot\left.\frac{\partial^{m-1}\left\{N_i[\phi_i(x,q)]-\varphi_i(x,q)\right\}}{\partial q^{m-1}}\right|_{q=0}, \tag{14}$$

and χ_m is defined by (11).

3 Implementation of Standard HAM and Modified HAM

For (3), let $L=I$ and $m=1$. Then (9) becomes

$$u_{i,1}(x)=\hbar R_{i,1}\left(\bar{u}_{i,0}\right), \tag{15}$$

where

$$R_{i,1}(\bar{u}_{i,0})=\left\{N_i[\phi_i(x,q)]-g_i(x)\right\}_{q=0}.$$

We can readily see that $\phi_i(x,q)|_{q=0}=\left[u_{i,0}(x)+\sum_{k=1}^{\infty}u_{i,k}(x)q^k\right]_{q=0}=u_{i,0}(x)$, then

$$R_{i,1}\left(\bar{u}_{i,0}\right)=u_{i,0}(x)-g_i(x)-\sum_{j=1}^{n}\left[\lambda_1\int_a^b k_{1,ij}(x,t)u_{j,0}(t)\,dt+\lambda_2\int_a^x k_{2,ij}(x,t)u_{j,0}(t)\,dt\right]. \tag{16}$$

Since $u_{i,0}(x)=g_i(x)$, from (15) and (16) it follows that

$$u_{i,1}(x)=-\hbar\sum_{j=1}^{n}\left[\lambda_1\int_a^b K_{1,ij}(x,t)g_j(t)\,dt+\lambda_2\int_a^x K_{2,ij}(x,t)g_j(t)\,dt\right]. \tag{17}$$

On the other hand, for $m>1$, we have

$$u_{i,m}(x)=u_{i,m-1}(x)+\hbar R_{i,m}\left(\bar{u}_{i,m-1}\right). \tag{18}$$

Let $m=2$, then, from (10) we obtain

$$\begin{aligned}R_{i,2}&=\left.\frac{\partial}{\partial q}\left[N_i\left[\phi_i(x,q)\right]-g_i(x)\right]\right|_{q=0}\\&=\frac{\partial}{\partial q}\left\{\phi_i(x,q)-g_i(x)-\sum_{j=1}^{n}\left[\lambda_1\int_a^b K_{1,ij}(x,t)\phi_j(x,q)\,dt\right.\right.\\&\qquad\left.\left.+\lambda_2\int_a^x K_{2,ij}(x,t)\phi_j(x,q)\,dt\right]\right\}_{q=0}.\end{aligned} \tag{19}$$

It is not difficult to see that $\frac{d}{dq}\phi_i(x,q)\Big|_{q=0} = u_{i,1}(x)$. Therefore, from (18) and (19), it follows that

$$u_{i,2}(x) = (1+\hbar)u_{i,1}(x) - \hbar\sum_{j=1}^{n}\left[\lambda_1\int_a^b K_{1,ij}(x,t)u_{j,1}(t)\,dt + \lambda_2\int_a^x K_{2,ij}(x,t)u_{j,1}(t)\,dt\right].$$

By continuing this procedure, we obtain

$$\begin{aligned} u_{i,m}(x) = (1+\hbar)u_{i,m-1}(x) - \hbar\sum_{j=1}^{n}\Bigg[\lambda_1\int_a^b K_{1,ij}(x,t)u_{j,m-1}(t)\,dt \\ + \lambda_2\int_a^x K_{2,ij}(x,t)u_{j,m-1}(t)\,dt\Bigg], m\geq 2. \end{aligned} \tag{20}$$

Eqs. (17) and (20) are the application of the standard HAM for problem (3).

Now, let us implement the MHAM for problem (3). To do this, the given function $g_i(x)$ is split into n terms:

$$g_i(x) = s_{i,0}(x) + s_{i,1}(x) + \cdots + s_{i,n}(x),$$

and the function $\varphi_i(x,q)$ is constructed in the form

$$\varphi_i(x,q) = s_{i,0}(x)q^0 + s_{i,1}(x)q^1 + \cdots + s_{i,n}(x)q^n = \sum_{j=0}^{n} q^j s_{ij}(x).$$

From (13) and (14), for $m=1$, it follows that

$$\begin{aligned} u_{i,1}(x) &= \hbar R_{i,1}\left[\bar{u}_{i,0}(x)\right] = \hbar\left\{N_i\left[\phi_i(x,q)\right] - \varphi_i(x,q)\right\}_{q=0} \\ &= \hbar\Bigg\{\phi_i(x,q) - \varphi_i(x,q) - \sum_{j=1}^{n}\Bigg[\lambda_1\int_a^b k_{1,ij}(x,t)\phi_j(t,q)\,dt \\ &\quad + \lambda_2\int_a^x k_{2,ij}(x,t)\phi_j(t,q)\,dt\Bigg]\Bigg\}_{q=0}. \end{aligned} \tag{21}$$

Since

$$\phi_i(x,q)|_{q=0} = \left[u_{i,0}(x) + \sum_{k=1}^{\infty} u_{i,k}(x)q^k\right]_{q=0} = u_{i,0}(x) = g_i(x),$$

$$\varphi_i(x,q)|_{q=0} = \left[s_{i,0}(x) + \sum_{j=1}^{n} q^j s_{i,j}(x)\right]_{q=0} = s_{i,0}(x),$$

upon substituting these into (21), we get

$$u_{i,1}(x) = \hbar\left\{g_i(x) - s_{i,0}(x) - \sum_{j=1}^{n}\left[\lambda_1\int_a^b k_{1,ij}(x,t)g_j(t)\,dt + \lambda_2\int_a^x k_{2,ij}(x,t)g_j(t)\,dt\right]\right\}. \tag{22}$$

Let $m = 2$. Then, from (13) and (14), and taking into account (11), we obtain

$$u_{i,2}(x) = u_{i,1}(x) + \hbar R_{i,2}\left[\bar{u}_{i,1}\right] = u_{i,1}(x) + \hbar\frac{\partial}{\partial q}\left[N_i\left[\phi_i(x,q)\right] - \varphi_i(x,q)\right]\Big|_{q=0}$$
$$= u_{i,1}(x) + \hbar\frac{\partial}{\partial q}\Bigg[\phi_i(x,q) - \lambda_1\int_a^b\sum_{j=1}^n k_{1,ij}(x,t)\phi_j(t,q)\,dt$$
$$- \lambda_2\int_a^x\sum_{j=1}^n k_{2,ij}(x,t)\phi_j(t,q)\,dt - \varphi_i(x,q)\Bigg]\Bigg|_{q=0}.$$

It is known that

$$\frac{\partial}{\partial q}\phi_i(x,q)\Big|_{q=0} = \frac{\partial}{\partial q}\left[u_{i,0}(x) + \sum_{k=1}^{\infty}q^k u_{i,k}(x)\right]\Bigg|_{q=0} = u_{i,1}(x),$$
$$\frac{\partial}{\partial q}\varphi_i(x,q)\Big|_{q=0} = \frac{\partial}{\partial q}\left[s_{i,0}(x) + \sum_{j=1}^{n}q^j s_{i,j}(x)\right]\Bigg|_{q=0} = s_{i,1}(x).$$

Taking these into account, we have

$$u_{i,2}(x) = (1+\hbar)\,u_{i,1}(x) - \hbar s_{i,1}(x) - \hbar\sum_{j=1}^n\left[\lambda_1\int_a^b k_{1,ij}(x,t)u_{j,1}(t)\,dt + \lambda_2\int_a^x k_{2,ij}(x,t)u_{j,1}(t)\,dt\right].$$

By continuing this procedure for any $i \in \{1,\dots,n\}$ and any $m \geq 1$, we get

$$\begin{aligned}
u_{i,1}(x) &= \hbar\left\{g_i(x) - s_{i,0}(x) - \sum_{j=1}^n\left[\lambda_1\int_a^b k_{1,ij}(x,t)g_j(t)\,dt + \lambda_2\int_a^x k_{2,ij}(x,t)g_j(t)\,dt\right]\right\}\\
u_{i,m}(x) &= (1+\hbar)\,u_{i,m-1}(x) - \hbar s_{i,m-1}(x) - \hbar\sum_{j=1}^n\Bigg[\lambda_1\int_a^b k_{1,ij}(x,t)u_{j,m-1}(t)\,dt\\
&\quad + \lambda_2\int_a^x k_{2,ij}(x,t)u_{j,m-1}(t)\,dt\Bigg], 2 \leq m \leq n+1,\\
u_{i,m}(x) &= (1+\hbar)u_{i,m-1}(x) - \hbar\sum_{j=1}^n\Bigg[\lambda_1\int_a^b k_{1,ij}(x,t)u_{j,m-1}(t)\,dt\\
&\quad + \lambda_2\int_a^x k_{2,ij}(x,t)u_{j,m-1}(t)\,dt\Bigg], m > n+1.
\end{aligned} \tag{23}$$

It should be emphasized that, for $m \geq 1$, the families $\{u_{i,m}(x)\}$ are governed by equations (9) and (13), and the series $\sum_{m=0}^{\infty} u_{i,m}$ converge to the exact solutions with different rates based on the choice of the initial guesses.

4 Gauss-Legendre Quadrature Formula

The Gauss-Legendre quadrature formula (GLQF) is a higher order quadrature formula given by Kythe & Schäferkotter [8] in the form

$$\int_{-1}^{1} f(\tau)\,d\tau = \sum_{i=1}^{n+1} w_i f(\tau_i) + R_{n+1}(f), \tag{24}$$

where

$$w_i = \frac{2}{(1-\tau_i^2)\left[P'_{n+1}(\tau_i)\right]^2}, \tag{25}$$

with $\sum_{i=1}^{n+1} w_i = 2$. It is exact for polynomials of degree $2n-1$ if weights w_i are defined by (25), and collocation points τ_i are chosen as the roots of Legendre polynomial $P_{n+1}(s)$, i.e.

$$P_{n+1}(\tau_i) = 0, \tag{26}$$

for $i \in \{1, 2, \dots, n+1\}$.

The error term of the GLQF (24) is given by

$$R_{n+1}(f) = \frac{2^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} f^{(2n+2)}(\xi),$$

for $-1 < \xi < 1$.

The GLQF (24) can be applied to the kernel integral on an interval $[a,b]$, as described by Eshkuvatov *et al.* [5]. For Fredholm and Volterra integral we have, respectively,

$$\begin{aligned} \int_a^b K_1(s,t)x(t)\,dt &= \frac{b-a}{2}\sum_{k=1}^{n+1} W_{1,k}(s)x(t_{1,k}) + R_n(x), \\ \int_a^s K_2(s,t)x(t)\,dt &= \frac{s-a}{2}\sum_{k=1}^{n+1} W_{2,k}(s)x(t_{2,k}) + R_n(x), \end{aligned} \tag{27}$$

where $s \in [a,b]$, and

$$\begin{aligned} W_{1,k}(s) &= k_1(s,t_{1,k})w_k, \qquad & t_{1,k} &= \frac{b-a}{2}r_k + \frac{b+a}{2}, \\ W_{2,k}(s) &= k_2(s,t_{2,k})w_k, \qquad & t_{2,k} &= \frac{s-a}{2}r_k + \frac{s+a}{2}, \end{aligned} \tag{28}$$

and r_k are the roots of the Legendre polynomial defined by (26). Eqs. (27) and (28) are crucial in computing the kernel integrals.

Using GLQF (27) and (28), we can obtain the standard HAM for $m \geq 2$ as follows

$$\begin{aligned} u_{i,1}(x) &= -\hbar \sum_{j=1}^{n} \left[\lambda_1 \frac{b-a}{2} \sum_{k=1}^{n+1} W_{1,kij}(x) g(t_{1,k}) + \lambda_2 \frac{x-a}{2} \sum_{k=1}^{n+1} W_{2,kij}(x) g(t_{2,k}) \right] \\ u_{i,m}(x) &= (1+\hbar) u_{i,m-1}(x) - \hbar \sum_{j=1}^{n} \left[\lambda_1 \frac{b-a}{2} \sum_{k=1}^{n+1} W_{1,kij}(x) u_{i,m-1}(t_{1,k}) \right. \\ &\quad \left. + \lambda_2 \frac{x-a}{2} \sum_{k=1}^{n+1} W_{2,kij}(x) u_{i,m-1}(t_{2,k}) \right], \; m \geq 2, \; i \in \{1, \dots, n\}. \end{aligned} \tag{29}$$

On the other hand, for the MHAM, we have for $i \in \{1, \dots, n\}$,

$$\begin{aligned} u_{i,1}(x) &= \hbar \left[g_i(x) - s_{i,0}(x) \right] - \hbar \sum_{j=1}^{n} \left[\lambda_1 \frac{b-a}{2} \sum_{k=1}^{n+1} W_{1,kij}(x) u_{i,0}(t_{1,k}) \right. \\ &\quad \left. + \lambda_2 \frac{x-a}{2} \sum_{k=1}^{n+1} W_{2,kij}(x) u_{i,0}(t_{2,k}) \right], \\ u_{i,m}(x) &= (1+\hbar) u_{i,m-1}(x) - \hbar s_{i,m-1}(x) - \hbar \sum_{j=1}^{n} \left[\lambda_1 \frac{b-a}{2} \sum_{k=1}^{n+1} W_{1,kij}(x) u_{i,m-1}(t_{1,k}) \right. \\ &\quad \left. + \lambda_2 \frac{x-a}{2} \sum_{k=1}^{n+1} W_{2,kij}(x) u_{i,m-1}(t_{2,k}) \right], 2 \leq m \leq n+1, \\ u_{i,m}(x) &= (1+\hbar) u_{i,m-1}(x) - \hbar \sum_{j=1}^{n} \left[\lambda_1 \frac{b-a}{2} \sum_{k=1}^{n+1} W_{1,kij}(x) u_{j,m-1}(t_{1,k}) \right. \\ &\quad \left. + \lambda_2 \frac{x-a}{2} \sum_{k=1}^{n+1} W_{2,kij}(x) u_{j,m-1}(t_{2,k}) \right], m > n+1, \end{aligned} \tag{30}$$

where

$$\begin{aligned} W_{1,kij}(x) &= k_{1,ij}(x, t_{1,k}) w_k, & t_{1,k} &= \frac{b-a}{2} r_k + \frac{b+a}{2}, \\ W_{2,k}(x) &= k_{2,ij}(x, t_{2,k}) w_k, & t_{2,k} &= \frac{x-a}{2} r_k + \frac{x+a}{2}, \end{aligned}$$

and r_k are the roots of Legendre polynomials defined by (26).

The MHAM with GLQF, as defined by (29) and (30), is very useful when the kernel integral has no analytic solution, and it also is highly accurate.

5 Numerical Experiments

In this section, we present a comparison between the MHAM and the standard HAM, as well as OHAM.

Example 1 (Ghazandi & Yari [6]): Let us consider the following system of Fredholm integral equations:

$$\begin{cases} u_1(x) = \frac{1}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3 + \int_0^1 (x-t)^3 u_1(t)\,dt + \int_0^1 (x-t)^2 u_2(t)\,dt, \\ u_2(x) = -\frac{1}{3}x^4 + \frac{23}{12}x^3 + \frac{3}{20}x^2 - \frac{41}{60}x - \frac{1}{30} + \int_0^1 (x-t)^4 u_1(t)\,dt + \int_0^1 (x-t)^3 u_2(t)\,dt. \end{cases}$$

Comparing with (3) we have $\lambda_1 = 1$, $\lambda_2 = 0$, and the kernels and functions are

$$\begin{aligned} k_{1,11}(x,t) &= (x-t)^3, \qquad k_{1,12}(x,t) = (x-t)^2, \\ k_{1,21}(x,t) &= (x-t)^4, \qquad k_{1,22}(x,t) = (x-t)^3, \\ g_1(x) &= \frac{1}{20} - \frac{11}{30}x + \frac{5}{3}x^2 - \frac{1}{3}x^3, \\ g_2(x) &= -\frac{1}{3}x^4 + \frac{23}{12}x^3 + \frac{3}{20}x^2 - \frac{41}{60}x - \frac{1}{30}. \end{aligned}$$

The exact solutions are

$$\begin{cases} u_1(x) = x^2, \\ u_2(x) = -x + x^2 + x^3. \end{cases}$$

As initial guess let us choose

$$\begin{cases} u_{1,0}(x) = g_1(x), \\ u_{2,0}(x) = g_2(x). \end{cases}$$

Then, using eq. (17) of the standard HAM, we obtain the first iteration as follows:

$$\begin{aligned} u_{1,1}(x) &= -\hbar\left[\int_0^1 (x-t)^3 g_1(t)\,dt + \int_0^1 (x-t)^2 g_2(t)\,dt\right] \\ &= -\hbar\left(\frac{61}{180}x^3 - \frac{161}{240}x^2 + \frac{11}{30}x - \frac{89}{1800}\right), \\ u_{2,1}(x) &= -\hbar\left[\int_0^1 (x-t)^4 g_1(t)\,dt + \int_0^1 (x-t)^3 g_2(t)\,dt\right] \\ &= -\hbar\left(\frac{61}{180}x^4 - \frac{133}{144}x^3 + \frac{41}{48}x^2 - \frac{1601}{5040}x + \frac{209}{6300}\right). \end{aligned}$$

Using eq. (20) of the standard HAM, the second and third iterations can be computed as follows:

$$\begin{aligned} u_{i,m}(x) = (1+\hbar)u_{i,m-1}(x) - \hbar\Bigg[&\int_0^1 k_{1,i1}(x,t)u_{1,m-1}(t)\,dt \\ &+ \int_0^1 k_{1,i2}(x,t)u_{2,m-1}(t)\,dt\Bigg], i \in \{1,2\}, m \geq 2. \end{aligned}$$

Now, let us solve the same problem with the MHAM. To do this, let us rewrite $g_1(x)$ and $g_2(x)$ as

$$
\begin{aligned}
g_1(x) &= \left(\frac{5}{3}x^2 - \frac{11}{30}x\right) + \left(-\frac{1}{3}x^3 + \frac{1}{20}\right) = s_{1,0}(x) + s_{1,1}(x),\\
g_2(x) &= \left(\frac{23}{12}x^3 - \frac{41}{60}x\right) + \left(-\frac{1}{3}x^4 + \frac{3}{20}x^2 - \frac{1}{30}\right) = s_{2,0}(x) + s_{2,1}(x).
\end{aligned}
$$

The corresponding functions $\varphi_i(x,q)$ can be written as

$$
\begin{aligned}
\varphi_1(x,q) &= s_{1,0}(x) + s_{1,1}(x)q,\\
\varphi_2(x,q) &= s_{2,0}(x) + s_{2,1}(x)q.
\end{aligned}
$$

Since

$$
\begin{aligned}
\phi_i(x,q)|_{q=0} &= u_{i,0}(x) = g_i(x),\\
\varphi_i(x,q)|_{q=0} &= s_{i,0}(x),
\end{aligned}
$$

for $i \in \{1,2\}$, then, for $m = 1$, eq. (22) of the MHAM gives us

$$
\begin{aligned}
u_{1,1}(x) &= \hbar\left\{g_1(x) - s_{1,0}(x) - \left[\int_0^1 (x-t)^3 g_1(t)\,dt + \int_0^1 (x-t)^2 g_2(t)\,dt\right]\right\}\\
&= -\hbar\left(\frac{121}{180}x^3 - \frac{161}{240}x^2 + \frac{11}{30}x - \frac{179}{1800}\right),\\
u_{2,1}(x) &= \hbar\left\{g_2(x) - s_{2,0}(x) - \left[\int_0^1 (x-t)^4 g_1(t)\,dt + \int_0^1 (x-t)^3 g_2(t)\,dt\right]\right\}\\
&= -\hbar\left(\frac{121}{180}x^4 - \frac{133}{144}x^3 + \frac{169}{240}x^2 - \frac{1601}{5040}x + \frac{419}{6300}\right).
\end{aligned}
$$

Eq. (23) gives any iterations of the MHAM as follows:

$$
u_{i,2}(x) = (1+\hbar)u_{i,1}(x) - \hbar s_{i,1}(x) - \hbar\left[\int_0^1 (x-t)^3 u_{1,1}(t)\,dt + \int_0^1 (x-t)^2 u_{2,1}(t)\,dt\right], i \in \{1,2\},
$$

$$
u_{i,m}(x) = (1+\hbar)u_{i,m-1}(x) - \hbar\left[\int_0^1 (x-t)^3 u_{1,m-1}(t)\,dt + \int_0^1 (x-t)^2 u_{2,m-1}(t)\,dt\right], i \in \{1,2\}.
$$

Tables 1 and 2 summarize results of standard HAM and MHAM, as well as results from other works.

Table 1: Numerical solutions of Example 1 for $\hbar = -1$ and $N = 3$. [$\star$] denotes data obtained from [6].

x	$\|u_1 - U_{1,3}\|$ HAM	$\|u_1 - U_{1,3}\|$ MHAM	$\|u_1 - U_{1,3}\|$ OHAM [$\star$]	$\|u_2 - U_{2,3}\|$ HAM	$\|u_2 - U_{2,3}\|$ MHAM	$\|u_2 - U_{2,3}\|$ OHAM [$\star$]
0.0	1.98×10^{-6}	3.10×10^{-4}	2.24×10^{-7}	1.32×10^{-6}	2.09×10^{-4}	1.54×10^{-7}
0.1	8.48×10^{-7}	2.02×10^{-4}	1.83×10^{-8}	4.24×10^{-7}	1.24×10^{-4}	1.75×10^{-8}
0.2	5.95×10^{-8}	1.17×10^{-4}	8.69×10^{-8}	2.78×10^{-8}	7.33×10^{-5}	9.08×10^{-8}
0.3	4.22×10^{-7}	5.28×10^{-5}	3.32×10^{-8}	1.77×10^{-7}	4.66×10^{-5}	8.76×10^{-8}
0.4	6.35×10^{-7}	4.68×10^{-6}	9.16×10^{-8}	1.51×10^{-7}	3.69×10^{-5}	3.18×10^{-8}
0.5	6.20×10^{-7}	3.04×10^{-5}	2.36×10^{-7}	6.20×10^{-8}	3.81×10^{-5}	5.04×10^{-8}
0.6	4.17×10^{-7}	5.57×10^{-5}	3.49×10^{-7}	6.61×10^{-9}	4.56×10^{-5}	1.30×10^{-7}
0.7	6.45×10^{-8}	7.48×10^{-5}	3.78×10^{-7}	6.50×10^{-8}	5.60×10^{-5}	1.77×10^{-7}
0.8	3.97×10^{-7}	9.08×10^{-5}	2.73×10^{-7}	3.02×10^{-7}	6.71×10^{-5}	1.56×10^{-7}
0.9	9.28×10^{-7}	1.07×10^{-4}	1.80×10^{-8}	7.65×10^{-7}	7.84×10^{-5}	3.33×10^{-8}
1.0	1.49×10^{-6}	1.28×10^{-4}	5.46×10^{-7}	1.49×10^{-6}	9.04×10^{-5}	2.31×10^{-7}

Table 2: Numerical solutions of Example 1 for $\hbar = -1$ and $N = 10$. [$\star$] denotes data obtained from [13].

x	$\|u_1 - U_{1,10}\|$ HAM	$\|u_1 - U_{1,10}\|$ MHAM	$\|u_1 - U_{1,10}\|$ HAM [$\star$]	$\|u_2 - U_{2,10}\|$ HAM	$\|u_2 - U_{2,10}\|$ MHAM	$\|u_2 - U_{2,10}\|$ HAM [$\star$]
0.0	1.71×10^{-17}	2.86×10^{-15}	–	3.02×10^{-18}	1.06×10^{-14}	–
0.1	3.82×10^{-17}	2.52×10^{-15}	1.11×10^{-15}	0.000	2.28×10^{-15}	4.09×10^{-16}
0.2	5.55×10^{-17}	7.34×10^{-15}	1.28×10^{-15}	2.78×10^{-17}	1.92×10^{-15}	2.43×10^{-16}
0.3	4.16×10^{-17}	1.12×10^{-14}	3.94×10^{-15}	0.000	2.83×10^{-15}	4.86×10^{-16}
0.4	5.55×10^{-17}	1.38×10^{-14}	8.33×10^{-15}	5.55×10^{-17}	1.55×10^{-15}	2.16×10^{-15}
0.5	5.55×10^{-17}	1.47×10^{-14}	3.33×10^{-15}	2.78×10^{-17}	6.24×10^{-16}	3.55×10^{-15}
0.6	0.000	1.34×10^{-14}	9.88×10^{-15}	8.33×10^{-17}	2.59×10^{-15}	5.88×10^{-15}
0.7	0.000	9.55×10^{-15}	2.28×10^{-15}	2.78×10^{-17}	2.64×10^{-15}	1.66×10^{-14}
0.8	3.33×10^{-16}	3.11×10^{-15}	1.80×10^{-14}	5.55×10^{-17}	9.99×10^{-16}	1.16×10^{-14}
0.9	2.22×10^{-16}	6.88×10^{-15}	2.22×10^{-16}	1.11×10^{-16}	9.88×10^{-15}	3.46×10^{-14}
1.0	2.22×10^{-16}	2.05×10^{-14}	1.38×10^{-14}	0.000	2.61×10^{-14}	1.86×10^{-14}

5.1 Convergence Interval for Example 1

The convergence interval can be obtain by imposing the following conditions:

$$\left.\frac{\partial \bar{u}_{iN}(x,\hbar)}{\partial x}\right|_{x=0} = f(h),\ h \in [a,b],\ i \in \{1,2\}. \tag{31}$$

In particular, for $N = 5$, (31) becomes

$$\left.\frac{\partial \bar{u}_{15}(x,\hbar)}{\partial x}\right|_{x=0} = -\frac{153188029}{423360000}h^5 - \frac{1047581}{576000}h^4 - \frac{55217}{15120}h^3 - \frac{3691}{1008}h^2 - \frac{11}{6}h - \frac{11}{30},$$

$$\left.\frac{\partial \bar{u}_{25}(x,\hbar)}{\partial x}\right|_{x=0} = \frac{16118857583}{50803200000}h^5 + \frac{89753269}{56448000}h^4 + \frac{1925171}{604800}h^3 + \frac{6415}{2016}h^2 + \frac{1601}{1008}h - \frac{41}{60}.$$

The graphs of these functions—called *ℏ-curves*—give the convergence region. Figure 1 shows the ℏ-curves on the interval $\hbar \in [-2.5, 0.5]$.

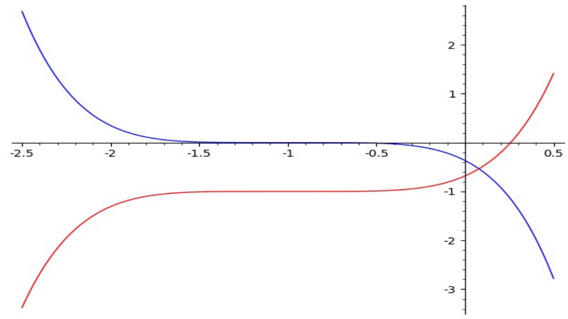

Figure 1: The ℏ-curves for Example 1 (HAM) for $N = 5$.

Example 2 (Shidfar & Molabahrami [13]): Consider the following system of linear Volterra integral equations of the second kind:

$$\begin{cases} u_1(x) = -x^3 - x + \displaystyle\int_0^x (u_2(t) + u_3(t))\ dt, \\ u_2(x) = \dfrac{1}{4}x^5 - \dfrac{1}{4}x^4 - \dfrac{1}{2}x^3 - 3x^2 - 1 + \displaystyle\int_0^x [(x-1)u_1(t) + tu_2(t) - xu_4(t)]\ dt, \\ u_3(x) = \dfrac{1}{2}x^6 - \dfrac{31}{6}x^3 + 2x^2 + 3 + \displaystyle\int_0^x \left[(x-t)u_1(t) - 3t^2u_4(t)\right]\ dt, \\ u_4(x) = x^3 - 5 + \displaystyle\int_0^x (2x - 3t)u_1(t)\ dt. \end{cases} \tag{32}$$

The exact solutions of this system are

$$\begin{cases} u_1(x) = x, & u_2(x) = x^2 - 1, \\ u_3(x) = 2x^2 + 3, & u_4(x) = x^3 - 5. \end{cases}$$

Comparing (32) with (3), we have $\lambda_1 = 0$, $\lambda_2 = 1$, and

$$\begin{aligned} g_1(x) &= -x - x^3, \\ g_2(x) &= \frac{1}{4}x^5 - \frac{1}{4}x^4 - \frac{1}{2}x^3 - 3x^2 - 1, \\ g_3(x) &= \frac{1}{2}x^6 - \frac{31}{6}x^3 + 2x^2 + 3, \\ g_4(x) &= x^3 - 5, \end{aligned}$$

with the kernels

$$\begin{array}{llll} K_{2,11}(x,t) = 0, & K_{2,12}(x,t) = 1, & K_{2,13}(x,t) = 1, & K_{2,14}(x,t) = 0, \\ K_{2,21}(x,t) = x - 1, & K_{2,22}(x,t) = t, & K_{2,23}(x,t) = 0, & K_{2,24}(x,t) = -x, \\ K_{2,31}(x,t) = x - t, & K_{2,32}(x,t) = 0, & K_{2,33}(x,t) = 0, & K_{2,34}(x,t) = -3t^2, \\ K_{2,41}(x,t) = 2x - 3t, & K_{2,42}(x,t) = 0, & K_{2,43}(x,t) = 0, & K_{2,44}(x,t) = 0. \end{array}$$

Let us choose as initial guess

$$u_{i,0}(x) = g_i(x),\, i \in \{1, 2, 3, 4\}.$$

Then, eqs. (17) and (20) of the standard HAM yields

$$u_{i,1}(x) = -\hbar \sum_{j=1}^{4} \int_0^1 K_{2,ij}(x,t) g_j(t)\, dt,$$

and

$$u_{i,m}(x) = (1+\hbar) u_{i,m-1} - \hbar \int_0^x K_{2,ij}(x,t) u_{j,m-1}(t)\, dt,$$

for $m \geq 2$ and $i \in \{1, 2, 3, 4\}$.

To apply eqs. (22) and (23) of the MHAM, let us decompose the functions $g_i(x)$ as follows:

$$\begin{aligned}
g_1(x) &= (-x) + (-x^3) = s_{1,0}(x) + s_{1,1}(x),\\
g_2(x) &= \left(-3x^2 - 1\right) + \left(\frac{1}{4}x^5 - \frac{1}{4}x^4 - \frac{1}{2}x^3\right) = s_{2,0}(x) + s_{2,1}(x),\\
g_3(x) &= \left(-\frac{31}{6}x^3 + 2x^2 + 3\right) + \left(\frac{1}{2}x^6\right) = s_{3,0}(x) + s_{3,1}(x),\\
g_4(x) &= (-5) + (x^3) = s_{4,0}(x) + s_{4,1}(x).
\end{aligned}$$

Then, the corresponding functions $\varphi_i(x,t)$ can be written as

$$\begin{aligned}
\varphi_1(x,t) &= s_{1,0}(x) + s_{1,1}(x) q, & \varphi_2(x,t) &= s_{2,0}(x) + s_{2,1}(x) q,\\
\varphi_3(x,t) &= s_{3,0}(x) + s_{3,1}(x) q, & \varphi_4(x,t) &= s_{4,0}(x) + s_{4,1}(x) q.
\end{aligned}$$

It is easy to see that for $i \in \{1, 2, 3, 4\}$

$$\begin{aligned}
\phi_1(x,q)|_{q=0} &= u_{i,0}(x) = g_i(x),\\
\varphi_1(x,q)|_{q=0} &= s_{i,0}(x).
\end{aligned}$$

Then, from (23) we obtain

$$\begin{aligned}
u_{i,1}(x) &= \hbar\left(g_i(x) - s_{i,0}(x) - \sum_{j=1}^{4} \int_0^x K_{2,ij}(x,t) g_j(t)\, dt \right),\\
u_{i,2}(x) &= (1+\hbar) u_{i,1}(x) - \hbar s_{i,1}(x) - \hbar \sum_{j=1}^{4} \int_0^x K_{2,ij}(x,t) u_{j,1}(t)\, dt,\\
u_{i,m}(x) &= (1+\hbar) u_{i,m-1}(x) - \hbar \sum_{j=1}^{4} \int_0^x K_{2,ij}(x,t) u_{j,m-1}(t)\, dt, m \geq 3.
\end{aligned}$$

Tables 3 and 4 summarize results of standard HAM and MHAM, as well as results from other works.

Table 3: Numerical solutions of $u_{1,10}(x)$ and $u_{2,10}(x)$ for Example 2 with $\hbar = -1$ and $N = 10$. $[\star]$ denotes [13].

x	$\|u_1 - U_{1,10}\|$ HAM	$\|u_1 - U_{1,10}\|$ MHAM	$\|u_1 - U_{1,10}\|$ HAM $[\star]$	$\|u_2 - U_{2,10}\|$ HAM	$\|u_2 - U_{2,10}\|$ MHAM	$\|u_2 - U_{2,10}\|$ HAM $[\star]$
0.0	0.000	0.000	–	0.000	0.000	0.000
0.1	5.55×10^{-17}	2.78×10^{-17}	2.14×10^{-15}	1.11×10^{-16}	1.11×10^{-16}	3.33×10^{-15}
0.2	3.89×10^{-16}	4.16×10^{-16}	2.56×10^{-14}	5.55×10^{-16}	3.33×10^{-16}	2.75×10^{-14}
0.3	2.04×10^{-14}	2.22×10^{-14}	3.98×10^{-14}	1.80×10^{-14}	1.79×10^{-14}	1.09×10^{-14}
0.4	2.58×10^{-13}	2.92×10^{-13}	3.57×10^{-13}	2.54×10^{-13}	2.52×10^{-13}	2.38×10^{-13}
0.5	1.96×10^{-12}	2.18×10^{-12}	2.45×10^{-12}	1.32×10^{-12}	1.29×10^{-12}	9.09×10^{-13}
0.6	1.27×10^{-11}	1.35×10^{-11}	1.30×10^{-11}	2.30×10^{-12}	1.78×10^{-12}	1.21×10^{-12}
0.7	6.34×10^{-11}	6.59×10^{-11}	5.77×10^{-11}	1.04×10^{-11}	1.48×10^{-11}	1.57×10^{-11}
0.8	1.96×10^{-10}	2.25×10^{-10}	1.48×10^{-10}	1.22×10^{-10}	1.44×10^{-10}	1.38×10^{-10}
0.9	1.52×10^{-10}	8.01×10^{-10}	1.38×10^{-10}	7.03×10^{-10}	5.32×10^{-10}	7.39×10^{-10}
1.0	2.75×10^{-9}	6.56×10^{-9}	4.19×10^{-9}	2.71×10^{-9}	2.87×10^{-9}	2.72×10^{-9}

Table 4: Numerical solutions of $u_{3,10}(x)$ and $u_{4,10}(x)$ for Example 2 with $\hbar = -1$ and $N = 10$. $[\star]$ denotes [13].

x	$\|u_3 - U_{3,10}\|$ HAM	$\|u_3 - U_{3,10}\|$ MHAM	$\|u_3 - U_{3,10}\|$ HAM $[\star]$	$\|u_4 - U_{4,10}\|$ HAM	$\|u_4 - U_{4,10}\|$ MHAM	$\|u_4 - U_{4,10}\|$ HAM $[\star]$
0.0	0.000	0.000	–	0.000	0.000	–
0.1	4.44×10^{-16}	4.44×10^{-16}	4.44×10^{-16}	8.88×10^{-16}	0.000	0.000
0.2	8.88×10^{-16}	0.000	2.22×10^{-15}	8.88×10^{-16}	0.000	0.000
0.3	1.33×10^{-15}	2.22×10^{-15}	4.88×10^{-15}	6.22×10^{-15}	4.44×10^{-15}	1.78×10^{-15}
0.4	6.66×10^{-14}	7.24×10^{-14}	7.82×10^{-14}	1.08×10^{-13}	1.04×10^{-13}	7.28×10^{-14}
0.5	8.87×10^{-13}	9.93×10^{-13}	1.02×10^{-12}	8.90×10^{-13}	8.40×10^{-13}	6.38×10^{-13}
0.6	7.56×10^{-12}	8.56×10^{-12}	8.67×10^{-12}	2.71×10^{-12}	2.13×10^{-12}	1.29×10^{-12}
0.7	5.54×10^{-11}	6.26×10^{-11}	5.90×10^{-11}	7.91×10^{-12}	1.40×10^{-11}	1.56×10^{-11}
0.8	3.63×10^{-10}	4.06×10^{-10}	3.62×10^{-10}	1.45×10^{-10}	2.09×10^{-10}	1.81×10^{-10}
0.9	1.92×10^{-9}	1.94×10^{-9}	1.83×10^{-9}	1.15×10^{-9}	1.92×10^{-9}	1.28×10^{-9}
1.0	8.03×10^{-9}	4.11×10^{-9}	7.25×10^{-9}	7.08×10^{-9}	1.54×10^{-8}	7.52×10^{-9}

5.2 Convergence Interval for Example 2

Figure 2 shows the four graphs of $\frac{\partial}{\partial x}\, u_{i,10}(x,h)\big|_{x=0}$, for $i \in \{1,2,3,4\}$, in blue, red, green, yellow, respectively. Note, graphs for red, green and yellow coincide with x-axis and cannot be clearly observed. From this we can see that convergence interval of the $\hbar$-curve is $[-8,6]$.

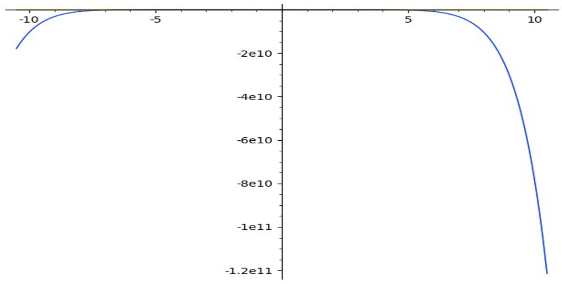

Figure 2: The $\hbar$-curve for Example 2 (HAM) for N=10.

6 Conclusions

In this paper, we have developed standard and modified HAM for solving systems of linear Fredholm-Volterra integral equations by combining Gauss-Legendre quadrature formulas. Numerical results showed that the developed method is comparable with OHAM in Ghazanfari & Yari [6] and with HAM developed in Shidfar & Molabahrami [13]. Tables 1–4 revealed that absolute minimum error is obtained in the most cases of modified HAM. The developed method can be used when integration problems cannot be solved analytically.

Acknowledgement This work was supported by Universiti Sains Islam Malaysia (USIM) under RMC Research Grant Scheme (FRGS, 2018), project code is USIM/ FRGS/ FST/ 055002/ 51118, and under National University of Uzbekistan, project code: OT-F4-28.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] S. Abbasbandy (2007). The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation. *Physics Letters A*, *361*(6), 478–483.

[2] M. B. Adam & J. A. Tawn (2011). Modification of Pickands' dependence function for ordered bivariate extreme value distribution. *Communications in Statistics - Theory and Methods*, *40*(9), 1687–1700.

[3] A. S. Bataineh, M. S. M. Noorani & I. Hashim (2009). Modified homotopy analysis method for solving systems of second-order BVPs. *Communications in Nonlinear Science and Numerical Simulation*, *14*(2), 430–442.

[4] M. M. Chawla & M. K. Jain (1968). Error estimates for Gauss quadrature formulas for analytic functions. *Mathematics of Computation*, *22*(101), 82–90.

[5] Z. K. Eshkuvatov, H. H. Hameed, B. M. Taib & N. M. A. Nik Long (2019). General 2×2 system of nonlinear integral equations and its approximate solution. *Journal of Computational and Applied Mathematics*, *361*, 528–546.

[6] B. Ghazanfari & N. Yari (2013). Optimal homotopy asymptotic method for solving system of Fredholm integral equations. *Communications in Numerical Analysis*, *2013*, Article ID: cna–00186, 15 pages. https://doi.org/10.5899/2013/cna-00186.

[7] T. Hayat, Z. Abbas, M. Sajid & S. Asghar (2007). The influence of thermal radiation on MHD flow of a second grade fluid. *International Journal of Heat and Mass Transfer*, *50*(5-6), 931–941.

[8] P. K. Kythe & M. R. Schäferkotter (2004). *Handbook of Computational Methods for Integration*. Chapman and Hall/CRC, New York.

[9] S. J. Liao (1992). *Proposed Homotopy Analysis Techniques for the Solution of Nonlinear Problems*. PhD thesis, Shanghai Jiaotong University, Shanghai.

[10] S. J. Liao (1995). An approximate solution technique which does not depend upon small parameters: A special example. *International Journal of Non-Linear Mechanics*, *30*(3), 371–380.

[11] S. J. Liao (1997). A kind of approximate solution technique which does not depend upon small parameters — II. an application in fluid mechanics. *International Journal of Non-Linear Mechanics*, *32*(5), 815–822.

[12] S. J. Liao (2005). Comparison between the homotopy analysis method and homotopy perturbation method. *Applied Mathematics and Computation*, *169*(2), 1186–1194.

[13] A. Shidfar & A. Molabahrami (2011). Solving a system of integral equations by an analytic method. *Mathematical and Computer Modelling*, *54*(1-2), 828–835.

[14] M. Turkyilmazoglu (2011). Some issues on HPM and HAM methods: A convergence scheme. *Mathematical and Computer Modelling*, *53*(9-10), 1929–1936.